Mechanics via Complex Numbers

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Introduction

Rotation seems to be the only way to make high school mechanics problems complicated to solve, therefore very important in high school physics contests. Traditionally, the concept of polar coordinate is adopted to describe rotation. However, I personally found that it is highly convenient to use the concept of complex number. This article is just to show a simple example. I will leave the rest of the expansion up to you. Have fun!

1 Motions of a Particle in Polar Coordinate

The position of a particle is described as

$$\vec{r} = r\hat{r} + \theta\hat{\theta}.$$

By using

$$\frac{d}{dt}\hat{\theta} = -\dot{\theta}\hat{r}$$

and

$$\frac{d}{dt}\hat{r} = r\dot{\theta}\hat{\theta},$$

which need to be verified by geometry, and relatively less mathematically rigorous, we have

$$\frac{d}{dt}\vec{r} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

and

$$\frac{d^2}{dt^2}\vec{r} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta}).$$

2 Motions of a Particle in Complex Plane

The position of a particle is elegantly described as

$$\vec{r} = re^{i\theta}.$$

By doing only basic calculus, we have

$$\frac{d}{dt}\vec{r} = \dot{r}e^{i\theta} + r\dot{\theta}ie^{i\theta}$$

and

$$\frac{d^2}{dt^2}\vec{r} = (\ddot{r} - r\dot{\theta}^2)e^{i\theta} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})ie^{i\theta},$$

which is identical to the result of polar coordinate.