Key Notes on Calculus for The fking Exams

Ray Chien

March 18, 2026

Contents

1 Functions

1.1 Inverse Functions

Inverse function exists **if and only if** the origin function is **one-to-one**. It can be tested by **horizontal line test** or by the definition of one-to-one function.

1.2 Domains of Inverse Trigonometric Functions

Table 1: Domains and Ranges of Inverse Trigonometric Functions

Function	**Domain**	**Range**	**Positive Functions**
$y = \sin^{-1}(x)$	$[-1, 1]$	$[-\frac{\pi}{2}, \frac{\pi}{2}]$	$\cos y, \sec y$
$y = \cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$	$\sin y, \csc y$
$y = \tan^{-1}(x)$	$(-\infty, \infty)$	$(-\frac{\pi}{2}, \frac{\pi}{2})$	$\cos y, \sec y$
$y = \cot^{-1}(x)$	$(-\infty, \infty)$	$(0, \pi)$	$\sin y, \csc y$
$y = \sec^{-1}(x)$	$\|x\| \geq 1$	$[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	$\tan x, -\cot x$
$y = \csc^{-1}(x)$	$\|x\| \geq 1$	$[0, \frac{\pi}{2}) \cup [\pi, \frac{3\pi}{2})$	$\tan x, -\cot x$

1.3 Exponential Law

$$e = a^{\frac{1}{\ln a}}$$

2 Limits

2.1 Limit Laws Do Not Apply to Composite Functions

The condition of

$$\lim_{x \to a} f(g(x)) = f\left(\lim_{x \to a} g(x)\right)$$

is that f is continuous at $\lim\limits_{x \to a} g(x)$.

2.2 Types of Functions that are continuous

- polynomials
- rational functions
- root functions
- trigonometric functions
- inverse trigonometric functions
- exponential functions
- logarithmic functions

2.3 Trigonometric Limits

$$\lim_{x \to 0} f(x) = 1$$
$$\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$$

2.4 Sketching A Curve

1. Find domain.
2. Find x- and y- intercepts.
3. Determine symmetry (about y-axis, about origin, or periodic).
4. Find asymptotes (vertical, horizontal, or slant).
5. Determine Increase or decrease by calculating f'.
6. Find Local maximum or minimum by calculating f'
7. Determine concavity and Inflection by calculating f''.
8. Sketch the curve.

3 Derivatives

3.1 Derivatives of Inverse Trigonometric Function

$$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\cos^{-1} x = \frac{-1}{\sqrt{1-x^2}}$$
$$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \qquad \frac{d}{dx}\cot^{-1} x = \frac{-1}{1+x^2}$$
$$\frac{d}{dx}\sec^{-1} x = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}\csc^{-1} x = \frac{-1}{x\sqrt{x^2-1}}$$

4 The Mean Value Theorem

4.1 Increasing and Decreasing

$$f(x_1) > f(x_2) \quad \text{whenever} \quad x_1 > x_2.$$

4.2 Maximum and Minimum

$f(c) \geq f(x)$ near c (local) or in the domain (absolute).

4.3 Critical Number

A critical number of f is a number c if $f'(c) = 0$ or $f'(c)$ does not exist. Endpoints of a closed interval are included.

4.4 Absolute Maximum And Minimum

If f is continuous on $[a, b]$ then it would be able to find the absolute maximum and minimum by comparing the function value on all the critical numbers.

5 Integral

5.1 A Sufficient Condition for Riemann Integrability

Let $f : [a, b] \to \mathbb{R}$ be a function defined on $[a, b]$, if f has no or only a finite number of jump discontinuities and removable discontinuities on and is continuous on the rest of the interval, then it is **Riemann Integrable** on $[a, b]$. If $x = c$ is a jump point, it means that the limits from both sides of c exist but $\lim\limits_{x\to c^-} f(x) \neq \lim\limits_{x\to c^+} f(x)$. If $x = d$ is a removable point, it means that $\lim\limits_{x\to d^-} f(x) = \lim\limits_{x\to d^+} f(x) \neq f(d)$. Note that The limits aforementioned does not consider as $\pm\infty$. Also, f must be bounded if the condition holds due to the Extreme Value Theorem.

6 Techniques of Integration

6.1 Substitution And Inverse Substitution

The complete statement of the Substitution Rule is: If $g'(x)$ is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))\, g(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

When applying trigonometric substitution, the hypotheses does not always hold. Here is an example: $\int_0^{2\pi} \frac{dx}{2+\cos x}$ using Weierstrass Substitution, that is $t = \tan\frac{x}{2}$. It is true that t is a function of x, but we know that the function derivative does not exist at $x = \pi \in [a, b]$. If we look at this inversely, x is not a function of i, which failed to have an exact one value of the "function". As the result, always be mindful of the hypotheses. For $\int_0^{2\pi} \frac{dx}{2+\cos x}$, we can divide the interval into $[0, \pi]$ and $[\pi, 2\pi]$, and evaluate with the theorem for improper integral. Always check if the substitution or inverse substitution can be written in the substitution form, that is $u = g(x)$.

6.2 Partial Fractions

Consider a rational function $f(x) = \dfrac{P(x)}{Q(x)}$ then follow the steps to find $\displaystyle\int f(x)\,dx$.

1. Divide $P(x)$ with $Q(x)$ and write $f(x) = S(x) + \dfrac{R(x)}{Q(x)}$. For $S(x)$, it is a polynomial and can be integrated effortlessly.

2. Factorize the denominator $Q(x)$ completely into factors of the form $(ax+b)^m$ and irreducible quadratic factors $(ax^2+bx+c)^n$, where $b^2 - 4ac < 0$.

$$Q(x) = C \cdot \prod_i (a_i x + b_i)^{m_i} \cdot \prod_j (\alpha_j x^2 + \beta_j x + \gamma_j)^{n_j}$$

3. Express $\dfrac{R(x)}{Q(x)}$ as a sum of partial fractions. The decomposition is determined by the factors of $Q(x)$:

   - **Case I: Linear Factors.** For each factor of the form $(ax+b)^m$, the partial fraction decomposition includes a sum of m terms:

$$\sum_{k=1}^{m} \frac{A_k}{(ax+b)^k} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_m}{(ax+b)^m}$$

   - **Case II: Irreducible Quadratic Factors.** For each factor of the form $(ax^2+bx+c)^n$, the decomposition includes a sum of n terms:

$$\sum_{k=1}^{n} \frac{B_k x + C_k}{(ax^2+bx+c)^k} = \frac{B_1 x + C_1}{ax^2+bx+c} + \cdots + \frac{B_n x + C_n}{(ax^2+bx+c)^n}$$

   where A_k, B_k, C_k are real constants to be determined.

4. Solve for the coefficient A_k, B_k, C_k with the following techniques:

   - By substituting x as all of the roots of the factors one by one, including the complex roots of quadratic factors, we can find all the coefficients over the largest power of each factor, that is A_m, B_n, C_n according to the description above.
   - To find the other coefficients $A_1, A_2, \ldots, A_{m-1}, B_1, B_2, \ldots, B_{n-1}, C_1, C_2, \ldots, C_{n-1}$, compare the coefficient or plug in convenient value of x.

5. Integrate every term. For terms with irreducible quadratic factors in the denominator $\dfrac{B_k x + C_k}{(ax^2+bx+c)^k}$, rewrite it in terms of $x + \dfrac{2b}{a}$.

6. Remember the integral

$$\int \frac{1}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c.$$

6.3 Priority of Integration by Parts

The preferred order of choosing u is **LIATE**, that is logarithmic, Inverse trigonometric, Algebraic, Trigonometric, Exponential functions. The core idea is to let u easy to be differentiated and v can be integrated without being more complicated, but the operation depends on actual situations.

6.4 Repeated Integration by Parts

Use "DI method," that is to list the derivatives and integrals of u and v.

Table 2: DI Method Table

D	**I**
u	$\frac{dv}{dx}$
u'	v
u''	$\int v\,dx$
$\vdots$	$\vdots$

The result of the is $uv - u'\int vdx + \ldots$. This approach is valid when $u^{(n)}$ would finally be 0, and v would not be more complicated but remain similar form when integrated, for instance, exponential and trigonometric function. Otherwise, it would also be valid if the product of the expression in a row only differs by a constant from that of the first row. It doesn't mean this is the only two ways to do repeated integration by parts, it is common to evaluate depends on the situation of every step.

6.5 Integrating Functions of Powers of Logarithmic function and Polynomials

Substitute $u = \ln x$ transfer into powers of exponential functions and polynomials, then integrate by parts.

6.6 Divergence And Convergence of The Improper integrals of Reciprocal Power Function

Table 3: Divergence And Convergence of Reciprocal Power Function

	$\int_0^1 \frac{1}{x^p}\,dx$	$\int_1^\infty \frac{1}{x^p}\,dx$
Convergent	p ¡ 1	p ¿ 1
Divergent	$p \geq 1$	$p \leq 1$

6.7 Integrations of Trigonometric Functions

1. $\int \sin^m x \, \cos^n x \, dx$, $m, n \in \mathbb{Z}$.

   (a) If m is odd, substitute $u = \cos x$:

   $$\int \sin^{2k+1} x \, \cos^n x \, dx = \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx.$$

   (b) If n is odd, substitute $u = \sin x$:

   $$\int \sin^m x \, \cos^{2k+1} x \, dx = \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx.$$

   (c) If n and m are both even, use the substitutions to express the function to integrate:

      i. If $|m - n| = 0 \vee 2$ and $mn \neq 0$, consider to apply $\sin x \, \cos x = \dfrac{1}{2} \sin 2x$.

      ii. $\sin^2 x = \dfrac{1 - \cos 2x}{2}$

      iii. $\cos^2 x = \dfrac{1 + \cos 2x}{2}$

   (d) For $\int \sin^m x \, dx$ and $\int \cos^n x \, dx$ with $m, n \geq 2$, Integrate by parts with $u = \sin^{m-1} x$ and $u = \cos^{n-1}$:

   $$\int \sin^m x \, dx = \frac{-1}{m} \sin^{m-1} \cos x + \frac{m-1}{m} \int \sin^{m-2} x \, dx$$

   $$\int \cos^n x \, dx = \frac{1}{n} \cos^{m-1} \sin x + \frac{n-1}{n} \int \cos^{n-2} x \, dx$$

   Although the method is valid for $m, n \in \mathbb{Z}$, it is better be applied when $m, n \geq 0$. Consider apply $\sin x \, \cos x = \dfrac{1}{2} \sin 2x$ when it is helpful.

2. $\int \tan^m x \, \sec^n x \, dx$, $m, n \in \mathbb{N}$.

(a) If m is odd, substitute $u = \sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\sec^2 - 1)^k \sec^{n-1} x \tan x \sec x \, dx.$$

(b) If n is even, substitute $u = \tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx.$$

(c) If m is even and n is odd, substitute $\tan^2 x$ with $\sec^2 x - 1$, and apply the reduction formula derived form integration by part with $u = \sec^{n-2} x$:

$$\int \sec^n x = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-1} x \, dx.$$

(d)

$$\int \tan^m x \, dx = \int \tan^{m-2} x \, (\sec^2 x - 1) \, dx = \frac{1}{m-1} \tan^{m-1} x - \int \tan^{m-2} x \, dx.$$

6.8 Weierstrass Substitution

The Weierstrass Substitution, which can be applied in integrations of complicated rational function of trigonometric functions, is a way to express the trigonometric functions by $t = \tan \frac{x}{2}$ in rational functions:

$$\sin\theta = \frac{2t}{1+t^2} \quad \cos\theta = \frac{1-t^2}{1+t^2} \quad \tan\theta = \frac{2t}{1-t^2}.$$

When applying the substitution $t = \tan \dfrac{\theta}{2}$, we have $d\theta = \dfrac{2}{1+t^2} \, dt$. Note that when using the substitution, it is not always valid to write $\theta = 2\tan^{-1} t$ without checking the interval of θ, so the intervals of this kind of integrations is a great challenge to deal with. By using this method, make sure the interval of integration for θ is originally between $\pm\pi$, otherwise, divide the interval into pieces.

6.9 Liouville's Theorem and Nonelementary Integrals

Not all integrals of elementary functions can be expressed in terms of elementary functions. Liouville's Theorem provides a criterion to determine solvability for certain forms.

Theorem (Liouville's Theorem for Exponential Integrals):
Let $f(x)$ and $g(x)$ be rational functions with $g(x)$ non-constant. If the integral

$$\int f(x) e^{g(x)} \, dx$$

is an elementary function, then it must be of the form

$$R(x) e^{g(x)} + C$$

where $R(x)$ is a rational function. If no such rational function $R(x)$ satisfies the differential equation derived from differentiation, the integral is **nonelementary**.

Common Nonelementary Integrals:
The following are famous examples of integrals that **cannot** be expressed using finite combinations of elementary functions:

- **Gaussian Integral:**
$$\int e^{-x^2}\,dx$$
- **Exponential Integral:**
$$\int \frac{e^x}{x}\,dx$$
- **Sine Integral:**
$$\int \frac{\sin x}{x}\,dx$$
- **Fresnel Integrals (Optics):**
$$\int \sin(x^2)\,dx \quad \text{and} \quad \int \cos(x^2)\,dx$$
- **Logarithmic Integral:**
$$\int \frac{1}{\ln x}\,dx$$
- **Elliptic Integrals (Arc length of an ellipse):**
$$\int \sqrt{1 - k^2 \sin^2 x}\,dx$$

7 Applications of Integration

7.1 Areas Between Curves

$$\int_a^b |f(x) - g(x)|\,dx$$

7.2 "Definition" of Volume

$$\int_a^b A(x)\,dx$$

7.3 Volume of a Solid of Revolution

Evaluating the volumes of solids of revolution about x axis.

1. Slide into washers or disks:

$$\int_a^b \pi \left[y_2^2 - y_1^2\right] \, dx$$

2. Slide into shells:

$$\int_c^d 2\pi y(x_2 - x_1) \, dy$$

7.4 Arc Length

If f' is continuous on $[a, b]$, then the length of the curve $y = f(x)$, $a \leq x \leq b$, is

$$\int_a^b \sqrt{1 + [f'(x)]^2} \, dx$$

7.5 Area of a Surface of Revolution

Evaluating the area of a surface of revolution of about x axis, where $\frac{dy}{dx}$ is continuous.

$$\int 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$$